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Fixed Point Theorems on Partial b-Metric Space using Fuzzy Mapping

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ABSTRACT: In the present work, we prove some fixed point theorems on partial b-metric space using fuzzy mapping for single and pair of mappings. These theorems extends the work of Shoaib *et al.* published in Fixed Point Theory and Application in 2018. One example is also given in the support of our result.

Keywords: Fixed Point Theorem, Fuzzy mapping, b-metric space, partial metric space

I. INTRODUCTION

Theory of the fixed point plays a key role in the various disciplines. This theory has many applications in pure as well as in applied mathematics. In 1922 Stefan Banach [1] proposed the idea of fixed point in contraction principle known as Banach Contraction Principle and it is most important and useful result in the fixed point theory. It got importance due to its application in the iterations work with the help of computer.

The fuzzy set notion was studied by Zadeh [2] in 1965. Then Butnariu [3] and Weiss [4] proposed the concept of fuzzy mapping and proved various fixed point results using this mapping. Later, the fuzzy contraction mapping introduced by Heilpern [5] and by using the fuzzy contraction mapping a fixed point theorem has been proved which is a fuzzy analogue of Nadler's[6] fixed point theorem for the multivalued mapping.

The generalization of usual metric space named as b-metric space was presented by the Bakhtin [7] and Czerwik [8] in 1989.After that many results are given in β-generalized weak contractive multi-functions and b-metric spaces [9-11].

Steve G Matthew [12] proposed the Partial metric space in 1994 and Banach contraction theorem was proved under the conditions of partial metric space.

In 2014, Satish Shukla [13] proposed the partial b-metric and using the Kannan type mapping, a fixed point theorem for Banach contraction was proved on this space.

In this present work, we introduce a fuzzy mapping in partial b-metric space and also some fixed point theorems are proved on the partial b-metric space using the fuzzy mapping for single as well as for the two mappings.

Definition 1.1[14] Let $Y = \emptyset$ and $t \geq 1$ and function $d'_b : Y \times Y \rightarrow [0, \infty)$ is b-metric when it satisfies the below mentioned properties:

(i) $d'_b(e_1', f_1') \geq 0 \quad \forall\, e_1', f_1' \in Y$
(ii) $d'_b(e_1', f_1') = 0 \; iff \; e_1' = f_1'$
(iii) $d'_b(e_1', f_1') = d'_b(f_1', e_1')$
(iv) $d'_b(e_1', f_1') \leq t[d'_b(e_1', i_1') + d'_b(i_1', f_1')] \quad \forall\, e_1', f_1', i_1' \in Y$

then the pair (Y, d'_b) is known as the b-metric space and it is the extension of usual metric space.

Definition 1.2[15] Let Y≠ ∅ and function $d'_p : Y \times Y \rightarrow [0, \infty)$ is known as partial-metric when it satisfies the below conditions:

(i) $e_1' = f_1' \Leftrightarrow d'_p(e_1', e_1') = d'_p(e_1', f_1') = d'_p(f_1', f_1') \;\; \forall\; e_1', f' \in Y$
(ii) $0 \leq d'_p(e_1', e_1') \leq d'_p(e_1', f_1')$
(iii) $d'_p(e_1', f_1') = d'_p(f_1', e_1')$
(iv) $d'_p(e_1', f_1') + d'_p(i_1', i_1') \leq \left[d'_p(e_1', i_1') + d'_p(i_1', f_1')\right]; \;\; \forall\; e_1', i_1', f_1' \in Y$

Then the pair(Y,d'_p) is known as partial metric space.

Definition 1.3[17] Let $Y \neq \emptyset$ and t be a positive integer and function $d'_{pb} : Y \times Y \rightarrow R^+$ is known as partial b-metric when it satisfies the following axioms:

(i) $e_1' = f_1' \Leftrightarrow d'_{pb}(e_1', e_1') = d'_{pb}(e_1', f_1') = d'_{pb}(f_1', f_1') \; \forall\; e_1', f_1' \in Y$
(ii) $d'_{pb}(e_1', e_1') \leq d'_{pb}(e_1', f_1')$
(iii) $d'_{pb}(e_1', f_1') = d'_{pb}(f_1', e_1')$
(iv) $d'_{pb}(e_1', f_1') + d'_{pb}(i_1', i_1') \leq t\left[d'_{pb}(e_1', i_1') + d'_{pb}(i_1', f_1')\right]; \; \forall\; e_1', f_1', i_1' \in Y$

then (Y,d'_{pb}) is known as the partial b-metric space. Then it is clear that class of d'_{pb} metric space is effectively larger than d'_p metric space and when we take t=1, $d'_{pb}$is a d'_bmetric space.

Then the each partial metric p on Y generates the topology $\tau_P$on X with base of the family of openp-balls{ $B_P(e_1', r') : e_1' \in Y, r' > 0\}$, where$B_P(e_1',r') = \{f_1' \in Y;\ d'_{pb}(e_1',f_1') < d'_{pb}(e_1',e_1') + r'\}$

Next we are giving one Example of Partial b Metric Space

Example 1.1 Let $Y = Q^+ \cup \{0\}$ and a function $d'_{pb}:Y \times Y \to Y$ by $d'_{pb}(r_1'',r_2'') = |r_1'' - r_2''|$ then we show that (Y, d'_{pb}) be partial b-metric space.

Proof: $d'_{pb}(r_1'',r_2'') = |r_1'' - r_2''|$

(i) $r_1'' = r_2''$

$$\Leftrightarrow d'_{pb}(r_1'',r_1'') = d'_{pb}(r_1'',r_2'') = d'_{pb}(r_2'',r_2'')$$

$$|r_1'' - r_1''| = |r_1'' - r_2''| = |r_2'' - r_2''|$$

$$0 = |r_1'' - r_2''| = 0$$

$$\Leftrightarrow r_1'' = r_2''$$

(ii) $d'_{pb}(r_1'',r_1'') = |r_1'' - r_1''|$

$$\leq |r_1'' - r_2''|$$

$$\leq d'_{pb}(r_1'',r_2'')$$

$$d'_{pb}(r_1'',r_1'') \leq d'_{pb}(r_1'',r_2'')$$

(iii) $d'_{pb}(r_1'',r_2'') = |r_1'' - r_2''|$

$$= |r_2'' - r_1''|$$

$$= d'_{pb}(r_2'',r_1'')$$

(iv) $d'_{pb}(r_1'',r_3'') = |r_1'' - r_3''|$

$$= |r_1'' - 2r_2'' + 2r_2'' - r_3''|$$

$$= |r_1'' - r_2'' + r_2'' - r_3'' + r_2'' - r_2''|$$

$$\leq |r_1'' - r_2''| + |r_2'' - r_3''| - |r_2'' - r_2''|$$

$$\leq t[|r_1'' - r_2''| + |r_2'' - r_3''|] - |r_2'' - r_2''| \quad as\ [t \geq 1]$$

$$d'_{pb}(r_1'',r_3'') \leq t\left[d'_{pb}(r_1'',r_2'') + d'_{pb}(r_2'',r_3'')\right] - d'_{pb}(r_2'',r_2'')$$

then, it satisfies all the conditions of partial b-metric space. So, (Y, d'_{pb}) is a partial b-metric space.

Definition 1.4[17] Let (Y,d'_{pb}) is partial b-metric space and let a sequence $\{y_n\}$ in Y and $y_1^* \in Y$. Then

(i) Sequence $\{y_n\}$ is convergent to y if$\lim_{n\to\infty} d'_{pb}(y_n, y_1^*)$ exists and it is finite. Here y_1^* is known as a d'_{pb}-limit of $\{y_n\}$.

(ii) In (Y,d'_{pb}) sequence $\{y_n\}$is a cauchy sequence if $\lim_{n,m\to\infty} d'_{pb}(y_n, y_m)$ exists and it is finite.

(iii) Let (Y,d'_{pb}) be the complete partial b-metric space if every Cauchy sequence $\{y_n\}$ in Y there $\exists\ y_1^* \in Y$ s.t.

$$\lim_{n,m\to\infty} d'_{pb}(y_n, y_m) = \lim_{n\to\infty} d'_{pb}(y_n, y_1^*) = d'_{pb}(y_1^*, y_1^*)$$

The limit of convergent sequence may not be unique in partial b-metric space.

Definition1.5 [15] Let A be subset of N is said to be closed in (N,p) if it is closed w.r.t.τ_P. Then A subset is known as bounded in(N, d'_{pb}) if there is $n_0 \in N$ and M > 0

Such that $a \in B_{Pb}(n_0, M) \forall a \in A$ i.e., $d'_{pb}(n_0, a) < d'_{pb}(n_0, n_0) + M\ \ \forall\ a \in A$.

Let CB^{Pb}(N) is collection of all closed, non-empty and bounded subsets of N w.r.t. the partial b-metricd'_{pb}.For$A \in CB^{Pb}(N)$,we define

$$d'_{pb}(x, A) = inf_{y_1^* \in A} d'_{pb}(x, y_1^*).$$

In A, there is atleast one best approximation if each $y_1^* \in N$, then A is said to be proximinal set. The set of all proximinal subsets of N is denoted by the P(N).

Definition1.6 [18] The function $H_{d'_{pb}}: P(N) \times P(N) \to [0,\infty)$, defined by

$$H_{d'_{pb}}(A,B) = max\{sup_{a\in A}\, d'_{pb}(a,B), sup_{b\in B}\, d'_{pb}(A,b)\}$$

is called partial Hausdroff b-metric space on P(N).

A function with domain N be fuzzy set in N and the values belongs to [0,1], then the group of all the fuzzy sets in N is F(N). If a fuzzy set K and $y_1^* \in Y$, then K(y_1^*) is function value which is known as the grade of membership of y_1^* in K. Thenα-level set of the fuzzy set is represented by $[K]_\alpha$ and defined as follows:

$[K]_\alpha = \{y_1^*: K(y_1^*) \geq \alpha\}$ where $\alpha \in (0,1]$

$$[K]_0 = \{y_1^*: K(y_1^*) \geq 0\}$$

Let $N \neq \emptyset$ and J is metric space. Then a mapping V is fuzzy mapping if mapping V from N into F(J). Fuzzy mapping V be the fuzzy subset on $N \times J$ with membership function V(y_1^*)(x) be grade of membership of x in V(y_1^*). Here, we represent the α-level set of V(y_1^*) as$[V(y_1^*)]_\alpha$. [16]

Definition1.7 [16] A point $y_1^* \in N$ is known as the fuzzy fixed point of the fuzzy mapping $T: N \to F(N)$ if $\exists\ \alpha \in (0,1]$, such that,$y_1^* \in [Ty_1^*]_\alpha$.

Lemma 1.1[18] Let two non-empty proximal subsets V and W of partial b-metric space(X, d'_{pb}) and if $v^* \in V$ then

$$d'_{pb}(v^*, W) \leq H(V, W)$$

Lemma 1.2[18] Let $\left(X, d'_{pb}\right)$be partial b-metric space and let $\left(P(N), H_{d'_{pb}}\right)$is partial Hausdorff b-metric space. Then, $\forall\ V, W \in P(N)$ and for each $v^* \in V$, there exists $w^*{}_{v^*} \in W$ satisfying

$$d'_{pb}(v^*, W) = d'_{pb}(v^*, w^*{}_{v^*})$$

then,

$$H_{d'_{pb}}(V, W) \geq\ d'_{pb}(a, w^*{}_{v^*})$$

II. MAIN RESULT

In this section, our purpose is to prove some fixed point theorems under fuzzy mapping for single as well as for two self-mappings.

Definition 2.1: Let (X, d'_{pb}) be partial b-metric space with the constant t ≥ 1 and a mapping V:X → F(X) is known as the multi valued generalized contraction if

$$H_{d'_{pb}}\left([Vx]_{\alpha(x)}, [Vy]_{\alpha(y)}\right) \leq a_1 d'_{pb}\left(x, [Vx]_{\alpha(x)}\right) + a_2 d'_{pb}\left(y, [Vy]_{\alpha(y)}\right) + a_3\begin{bmatrix} d'_{pb}\left(x, [Vy]_{\alpha(y)}\right) + \\ d'_{pb}\left(y, [Vx]_{\alpha(x)}\right) \end{bmatrix} + \frac{a_4}{3t}\left[\frac{\left\{d'_{pb}\left(x, [Vy]_{\alpha(y)}\right) + d'_{pb}\left(y, [Vx]_{\alpha(x)}\right)\right\} d'_{pb}\left(y, [Vy]_{\alpha(y)}\right)}{d'_{pb}(x,y) + d'_{pb}\left(y, [Vy]_{\alpha(y)}\right)}\right] + \frac{a_5}{2t} d'_{pb}(x,y)$$

for all $x, y \in X$ and $a_i \geq 0$, i = 1,2,...5 with $ta_1 + a_2 + t(t+1)a_3 + \frac{a_4}{3} + \frac{a_5}{2} < 1$.

Theorem 2.1 Let (Y, d'_{pb}) be the complete partial b-metric space with constant $t \geq 1$. Let $V: Y \to F(Y)$ is fuzzy mapping, and let m_0 is any arbitrary point in Y. Let there $\exists\, \alpha(m) \in (0,1]\ \forall m \in Y$ and V satisfies the following conditions:

$$H_{d'_{pb}}([Vm]_{\alpha(m)}, [Vn]_{\alpha(n)}) \leq a_1 d'_{pb}(m, [Vm]_{\alpha(m)}) + a_2 d'_{pb}(n, [Vn]_{\alpha(n)}) + a_3 \begin{bmatrix} d'_{pb}(m, [Vn]_{\alpha(n)}) \\ + d'_{pb}(n, [Vm]_{\alpha(m)}) \end{bmatrix} + \frac{a_4}{3t}\left[\frac{\{d'_{pb}(m, [Vn]_{\alpha(n)}) + d'_{pb}(n, [Vm]_{\alpha(m)})\}\, d'_{pb}(n, [Vn]_{\alpha(n)})}{d'_{pb}(m,n) + d'_{pb}(n, [Vn]_{\alpha(n)})}\right] + \frac{a_5}{2t} d'_{pb}(m,n)$$

And $d'_{pb}(m_0, [Vm_0]_{\alpha(m_0)}) \leq v(1 - tv)r\ \forall m, n \in \overline{B_{d'_{pb}}(m_0, r)}$, $r > 0$ and $tv < 1$,

$$where\ v = \frac{\left(a_1 + ta_3 + \frac{a_5}{2t}\right)}{\left(1 - \left(a_2 + ta_3 + \frac{a_4}{3}\right)\right)}.$$

Also, $a_i \geq 0$, where $i = 1,2 \ldots.5$ with $ta_1 + a_2 + t(t+1)a_3 + \frac{a_4}{3} + \frac{a_5}{2} < 1$ and $a_1 + a_2 + 2a_3 + \frac{a_4}{3} + \frac{a_5}{2} \leq 1$ where $\sum_{i=1}^{5} a_i < 1$. Then, there exist m^* in $\overline{B_{d'_{pb}}(m_0, r)}$ such that $m^* \in [Vm^*]_{\alpha(m^*)}$.

Proof: Let m_0 be any arbitrary point in Y such that $m_1 \in [Vm_0]_{\alpha(m_0)}$. Let a sequence $\{m_l\}$ of points in Y such that $m_l \in [Vm_{l-1}]_{\alpha(m_{l-1})}$.

First, we show that $m_l \in \overline{B_{d'_{pb}}(m_0, r)}\ \ \forall n \in N$

$$d'_{pb}(m_0, m_1) = d'_{pb}(m_0, [Vm_0]_{\alpha(m_0)}) \leq v(1 - tv)r < r$$

$$\Rightarrow m_1 \in \overline{B_{d'_{pb}}(m_0, r)}. Let\ m_2, m_3, \ldots \ldots m_j \in \overline{B_{d'_{pb}}(m_0, r)}, j \in N.$$

Now, by using lemma 1.1 we get

$$d'_{pb}(m_j, m_{j+1}) = H_{d'_{pb}}\left([Vm_{j-1}]_{\alpha(m_{j-1})}, [Vm_j]_{\alpha(m_j)}\right)$$
$$\leq a_1 d'_{pb}\left(m_{j-1}, [Vm_{j-1}]_{\alpha(m_{j-1})}\right) + a_2 d'_{pb}\left(m_j, [Vm_j]_{\alpha(m_j)}\right) + a_3 \begin{bmatrix} d'_{pb}\left(m_{j-1}, [Vm_j]_{\alpha(m_j)}\right) + \\ d'_{pb}\left(m_j, [Vm_{j-1}]_{\alpha(m_{j-1})}\right) \end{bmatrix}$$
$$+ \frac{a_4}{3t}\left[\frac{\left\{d'_{pb}\left(m_{j-1}, [Vm_j]_{\alpha(m_j)}\right) + d'_{pb}\left(m_j, [Vm_{j-1}]_{\alpha(m_{j-1})}\right)\right\} d'_{pb}\left(m_j, [Vm_j]_{\alpha(x_j)}\right)}{d'_{pb}(m_{j-1}, m_j) + d'_{pb}\left(m_j, [Vm_j]_{\alpha(m_j)}\right)}\right] + \frac{a_5}{2t} d'_{pb}(m_{j-1}, m_j)$$
$$\leq a_1 d'_{pb}(m_{j-1}, m_j) + a_2 d'_{pb}(m_j, m_{j+1}) + a_3 \begin{bmatrix} d'_{pb}(m_{j-1}, m_{j+1}) + \\ d'_{pb}(m_j, m_j) \end{bmatrix} + \frac{a_4}{3t}\left[\frac{\{d'_{pb}(m_{j-1}, m_{j+1}) + d'_{pb}(m_j, m_j)\}\, d'_{pb}(m_j, m_{j+1})}{d'_{pb}(m_{j-1}, m_j) + d'_{pb}(m_j, m_{j+1})}\right] + \frac{a_5}{2t} d'_{pb}(m_{j-1}, m_j)$$

Then by using 4th condition of partial b-metric space, we get

$$d'_{pb}(m_j, m_{j+1}) \leq a_1 d'_{pb}(m_{j-1}, m_j) + a_2 d'_{pb}(m_j, m_{j+1}) + a_3 \begin{bmatrix} t\{d'_{pb}(m_{j-1}, m_j) + \\ d'_{pb}(m_j, m_{j+1})\} - \\ d'_{pb}(m_j, m_j) + d'_{pb}(m_j, m_j) \end{bmatrix} + \frac{a_4}{3t}\left[\frac{\{d'_{pb}(m_{j-1}, m_{j+1}) + d'_{pb}(m_j, m_j)\}\, d'_{pb}(m_j, m_{j+1})}{d'_{pb}(m_{j-1}, m_j) + d'_{pb}(m_j, m_j)}\right] + \frac{a_5}{2t} d'_{pb}(m_{j-1}, m_j)$$

$$d'_{pb}(m_j, m_{j+1}) \leq a_1 d'_{pb}(m_{j-1}, m_j) + a_2 d'_{pb}(m_j, m_{j+1}) + a_3 \begin{bmatrix} td'_{pb}(m_{j-1}, m_j) \\ + td'_{pb}(m_j, m_{j+1}) \end{bmatrix} + \frac{a_4}{3} d'_{pb}(m_j, m_{j+1}) + \frac{a_5}{2t} d'_{pb}(m_{j-1}, m_j)$$

$$d'_{pb}(m_j, m_{j+1}) \leq \left(a_1 + ta_3 + \frac{a_5}{2t}\right) d'_{pb}(m_{j-1}, m_j) + \left(a_2 + ta_3 + \frac{a_4}{3}\right) d'_{pb}(m_j, m_{j+1})$$

$$\left(1 - \left(a_2 + ta_3 + \frac{a_4}{3}\right)\right) d'_{pb}(m_j, m_{j+1}) \leq \left(a_1 + ta_3 + \frac{a_5}{2t}\right) d'_{pb}(m_{j-1}, m_j)$$

$$d'_{pb}(m_j, m_{j+1}) \leq \frac{\left(a_1 + ta_3 + \frac{a_5}{2t}\right)}{\left(1 - \left(a_2 + ta_3 + \frac{a_4}{3}\right)\right)} d'_{pb}(m_{j-1}, m_j)$$

$$d'_{pb}(m_j, m_{j+1}) \leq v d'_{pb}(m_{j-1}, m_j)$$

Where $v = \frac{\left(a_1 + ta_3 + \frac{a_5}{2t}\right)}{\left(1 - \left(a_2 + ta_3 + \frac{a_4}{3}\right)\right)} < 1$

On the same way, we get

$$d'_{pb}(m_j, m_{j+1}) \leq v^j d'_{pb}(m_0, m_1) \quad (1)$$

Now,

$$d'_{pb}(m_0, m_{j+1}) \leq \begin{bmatrix} td'_{pb}(m_0, m_1) + \\ t^2 d'_{pb}(m_1, m_2) + \cdots \\ + t^{j+1} d'_{pb}(m_j, m_{j+1}) \end{bmatrix} - d'_{pb}(m_1, m_1) - d'_{pb}(m_2, m_2) - \cdots - d'_{pb}(m_j, m_j)$$

$$d'_{pb}(m_0, m_{j+1}) \leq \begin{bmatrix} td'_{pb}(m_0, m_1) + \\ t^2 v d'_{pb}(m_0, m_1) + \cdots \\ + t^{j+1} v^j d'_{pb}(m_0, m_1) \end{bmatrix} - \sum_{h=1}^{j} d'_{pb}(m_h, m_h) \quad (2)$$

Now, we show that $d'_{pb}(m_h, m_h) = 0$

Then,

$$d'_{pb}(m_h, m_h) = H_{d'_{pb}}([Vm_{h-1}]_{\alpha(m_{h-1})}, [Vm_{h-1}]_{\alpha(m_{h-1})}) \leq a_1 d'_{pb}(m_{h-1}, [Vm_{h-1}]_{\alpha(m_{h-1})})$$

$$+a_2 d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)$$
$$+a_3\begin{bmatrix} d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)+ \\ d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)\end{bmatrix}$$
$$+\frac{a_4}{3t}\begin{bmatrix}\dfrac{\begin{Bmatrix} d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)+ \\ d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)\end{Bmatrix} d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)}{d'_{pb}(m_{h-1},m_{h-1})+ \\ d'_{pb}\left(m_{h-1},[Vm_{h-1}]_{\alpha(m_{h-1})}\right)}\end{bmatrix}$$
$$+\frac{a_5}{2t}d'_{pb}(m_{h-1},m_{h-1})$$
$$\leq a_1 d'_{pb}(m_{h-1},m_h)+a_2 d'_{pb}(m_{h-1},m_h)$$
$$+a_3\begin{bmatrix} d'_{pb}(m_{h-1},m_h)+ \\ d'_{pb}(m_{h-1},m_h)\end{bmatrix}$$
$$+\frac{a_4}{3t}\begin{bmatrix}\dfrac{\{d'_{pb}(m_{h-1},m_h)+d'_{pb}(m_{h-1},m_h)\} d'_{pb}(m_{h-1},m_h)}{d'_{pb}(m_{h-1},m_{h-1})+d'_{pb}(m_{h-1},m_h)}\end{bmatrix}$$
$$+\frac{a_5}{2t}d'_{pb}(m_{h-1},m_{h-1})$$
$$\leq (a_1+a_2+2a_3)d'_{pb}(m_{h-1},m_h)$$
$$+\frac{a_4}{3t}\begin{bmatrix}\dfrac{\{2d'_{pb}(m_{h-1},m_h)\} d'_{pb}(m_{h-1},m_h)}{d'_{pb}(m_{h-1},m_{h-1})+ \\ d'_{pb}(m_{h-1},m_h)}\end{bmatrix}$$
$$+\frac{a_5}{2t}d'_{pb}(m_{h-1},m_{h-1})$$

Then by 1[st] condition of partial b-metric space

$$d'_{pb}(m_h,m_h)\leq (a_1+a_2+2a_3)d'_{pb}(m_{h-1},m_h)$$
$$+\frac{a_4}{3t}\begin{bmatrix}\dfrac{\{2d'_{pb}(m_{h-1},m_h)\} d'_{pb}(m_{h-1},m_h)}{d'_{pb}(m_{h-1},m_h)+ \\ d'_{pb}(m_{h-1},m_h)}\end{bmatrix}$$
$$+\frac{a_5}{2t}d'_{pb}(m_{h-1},m_h)$$

$$\leq (a_1+a_2+2a_3)d'_{pb}(m_{h-1},m_h)$$
$$+\frac{a_4}{3t}\begin{bmatrix}\dfrac{\{2d'_{pb}(m_{h-1},m_h)\} d'_{pb}(m_{h-1},m_h)}{2d'_{pb}(m_{h-1},m_h)}\end{bmatrix}$$
$$+\frac{a_5}{2t}d'_{pb}(m_{h-1},m_h)$$

$$\leq (a_1+a_2+2a_3)d'_{pb}(m_{h-1},m_h)$$
$$+\frac{a_4}{3t}d'_{pb}(m_{h-1},m_h)$$
$$+\frac{a_5}{2t}d'_{pb}(m_{h-1},m_h)$$
$$\leq \left(a_1+a_2+2a_3+\frac{a_4}{3}+\frac{a_5}{2}\right)d'_{pb}(m_{h-1},m_h)$$

$d'_{pb}(m_h,m_h)\leq \lambda d'_{pb}(m_{h-1},m_h)$

$d'_{pb}(m_h,m_h)\leq \lambda^2 d'_{pb}(m_{h-2},m_{h-1})$

and so on.

$d'_{pb}(m_h,m_h)\leq \lambda^h d'_{pb}(m_0,m_1)$

Where$\lambda=a_1+a_2+2a_3+\frac{a_4}{3}+\frac{a_5}{2}$

Then by taking limit $h\to\infty$, we get

$d'_{pb}(m_h,m_h)=0$

Now, from (2) we get

$$d'_{pb}\left(m_0,m_{j+1}\right)\leq\begin{bmatrix} td'_{pb}(m_0,m_1)+ \\ t^2 v d'_{pb}(m_0,m_1)+\cdots\ldots\ldots+ \\ t^{j+1}v^j d'_{pb}(m_0,m_1)\end{bmatrix}$$

$$d'_{pb}\left(m_0,m_{j+1}\right)\leq\frac{t(1-(tv)^j)}{1-tv}d'_{pb}(m_0,m_1)$$
$$d'_{pb}\left(m_0,m_{j+1}\right)\leq\frac{t\left(1-(tv)^j\right)}{1-tv}v(1-tv)r<r$$

$\Rightarrow m_{j+1}\in\overline{B_{d'_{pb}}(m_0,r)}$. Hence by induction $m_l\in\overline{B_{d'_{pb}}(m_0,r)}$ $for\ all\ l\in N$.

Now, we show that sequence $\{m_l\}$ is a Cauchy sequence.

So, equation (1) can be written as

$d'_{pb}(m_l,m_{l+1})\leq v^{l-1}d'_{pb}(m_0,m_1)$

Let i and l be two positive integers, $i<l$

$$d'_{pb}(m_i,m_l)\leq t[d'_{pb}\begin{pmatrix} m_i,m_{i+1})+ \\ d'_{pb}(m_{i+1},m_l\end{pmatrix}]$$
$$-d'_{pb}(m_{i+1},m_{i+1})$$
$$d'_{pb}(m_i,m_l)\ \leq td'_{pb}(m_i,m_{i+1})+ t^2 d'_{pb}(m_{i+1},m_{i+2})+\cdots+ t^{l-i}d'_{pb}(m_{l-1},m_l)$$
$$d'_{pb}(m_i,m_l)\leq\ [tv^i d'_{pb}(m_0,m_1)+ t^2 v^{i+1}d'_{pb}(m_0,m_1)+\cdots+ t^{l-i}v^{l-1}d'_{pb}(m_0,m_1)]$$
$$d'_{pb}(m_i,m_l)\leq tv^i\begin{bmatrix}1+tv+t^2v^2+\cdots+ \\ t^{l-i-1}v^{l-i-1}\end{bmatrix}d'_{pb}(m_0,m_1)$$
$$d'_{pb}(m_i,m_l)\ \ \leq\frac{tv^i}{1-vt}d'_{pb}(m_0,m_1)$$
$$\therefore\ d'_{pb}(m_i,m_l)\leq\frac{tv^i}{1-vt}d'_{pb}(m_0,m_1)$$

As $d'_{pb}(m_i,m_l)\to 0\ as\ i,l\to\infty$

Hence, sequence $\{m_l\}$ is Cauchy sequence in Y.

By completeness, there exist a point m^* which belongs to Y.

Since, $\{m_l\}$ converges to m^* as $l\to\infty$

$\lim_{l\to\infty}d'_{pb}(m_l,m^*)=d'_{pb}(m^*,m^*)=$
$\lim_{l\to\infty}d'_{pb}(m_l,m_l)=0$

(3)

Now, we show that m^* be the fixed point of T.

$$d'_{pb}\left(m^*,[Vm^*]_{\alpha(m^*)}\right)\leq$$
$$t\begin{bmatrix} d'_{pb}(m^*,m_{l+1}) \\ +d'_{pb}\left(m_{l+1},[Vm^*]_{\alpha(m^*)}\right)\end{bmatrix}$$
$$-d'_{pb}(m_{l+1},m_{l+1})$$
$$d'_{pb}\left(m^*,[Vm^*]_{\alpha(m^*)}\right)\leq$$
$$td'_{pb}\left(m^*,[Vm_l]_{\alpha(m_l)}\right)+$$
$$tH_{d'_{pb}}\left([Vm_l]_{\alpha(m_l)},[Vm^*]_{\alpha(m^*)}\right)$$
$$d'_{pb}\left(m^*,[Vm^*]_{\alpha(m^*)}\right)\leq td'_{pb}\left(m^*,[Vm_l]_{\alpha(m_l)}\right)$$

$$+t\left\{\begin{matrix} a_1 d'_{pb}(m_l,[Vm_l]_{\alpha(m_l)})+ \\ a_2 d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})+ \\ a_3\begin{bmatrix} d'_{pb}(m_l,[Vm^*]_{\alpha(m^*)})+ \\ d'_{pb}(m^*,[Vm_l]_{\alpha(m_l)}) \end{bmatrix}+ \\ \frac{a_4}{3t}\left[\frac{\begin{Bmatrix} d'_{pb}(m_l,[Vm^*]_{\alpha(m^*)})+ \\ d'_{pb}(m^*,[Vm_l]_{\alpha(m_l)}) \end{Bmatrix} d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})}{d'_{pb}(m_l,m^*)+d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})}\right]+ \\ \frac{a_5}{2t}d'_{pb}(m_l,m^*) \end{matrix}\right\}$$

$$d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})\le t d'_{pb}(m^*,m_{l+1})$$
$$+t\left\{\begin{matrix} a_1 d_{pb}(m_l,m_{l+1})+ \\ d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})+ \\ a_3\begin{bmatrix} d'_{pb}(x_l,[Vx^*]_{\alpha(x^*)})+ \\ d'_{pb}(m^*,m_{l+1}) \end{bmatrix}+ \\ \frac{a_4}{3t}\left[\frac{\{d'_{pb}(m_l,[Tm^*]_{\alpha(m^*)})+d'_{pb}(m^*,m_{l+1})\} d'_{pb}(m^*,[Tm^*]_{\alpha(m^*)})}{d'_{pb}(m_l,m^*)+d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})}\right]+ \\ \frac{a_5}{2t}d'_{pb}(m_l,m^*) \end{matrix}\right\}$$

$$d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})\le t d'_{pb}(m^*,m_{l+1})$$
$$+t\left\{\begin{matrix} a_1 d'_{pb}(m^*,m_{l+1})+ \\ a_2 d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})+ \\ a_3\begin{bmatrix} d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})+ \\ d'_{pb}(m^*,m_{l+1}) \end{bmatrix}+ \\ \frac{a_4}{3t}\left[\frac{\{d'_{pb}(m_l,[Vm^*]_{\alpha(m^*)})+d'_{pb}(m^*,m^*)\} d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})}{d'_{pb}(m_l,m^*)+d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})}\right]+ \\ \frac{a_5}{2t}d'_{pb}(m_l,m^*) \end{matrix}\right\}$$

[as $m_l \to m^*$]

$$d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})\le (t+ta_1+ta_3)d'_{pb}(m^*,m_{l+1}) + \left(ta_2+ta_3+\frac{a_4}{3}\right)d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)}) + \frac{a_5}{2}d'_{pb}(m_l,m^*)$$

By using eq. (3) and taking limit $l\to\infty$, *we get*

$$\left[1-\left(ta_2+ta_3+\frac{a_4}{3}\right)\right]d'_{pb}(m^*,[Vm^*]_{\alpha(m^*)})\le 0$$

So, we get

$$m^*\in[Vm^*]_{\alpha(m^*)}$$

Hence, m^* be the fixed point of V.

Example 2.1: Let $Y=Q^+\cup\{0\}$ and $d'_{pb}=|r_1''-s_1''|$, whenever $r_1'',s_1''\in Y$ then (Y,d'_{pb}) be complete partial b-metric space with constant $t>1$. Define a fuzzy mapping $V:Y\to F(Y)$ by

$$V(r_1'')(v)=\begin{cases} 0, & 0\le v\le r_1''/4 \\ 1/2, & r_1''/4<v\le r_1''/3 \\ 1/4, & r_1''<v\le r_1''/2 \\ 1, & r_1''/2<v\le 1 \end{cases}$$

For all $r_1''\in Y$, *there* $\exists\,\alpha(r_1'')=1$ s.t.

$$[Vr_1'']_{\alpha(r_1'')}=\left[0,\frac{r_1''}{4}\right]$$

Consider $r_1''=1\in X$ and $r=4$, then $\overline{B_{d'_{pb}}(r_1'',r)}=[0,5]$.

Let $a_1=\frac{1}{50}, a_2=\frac{1}{40}, a_3=\frac{1}{30}, a_4=\frac{1}{20}, a_5=\frac{1}{10}$. Then

$$H_{d'_{pb}}([Vr_1'']_{\alpha(r_1'')},[Vs_1'']_{\alpha(s_1'')}) \le \frac{1}{50}\left|r_1''-\frac{r_1''}{4}\right|+\frac{1}{40}\left|s_1''-\frac{s_1''}{4}\right| + \frac{1}{30}\begin{bmatrix}\left|r_1''-\frac{s_1''}{4}\right|+ \\ \left|s_1''-\frac{r_1''}{4}\right|\end{bmatrix} + \frac{1}{60t}\left[\frac{\left\{\left|r_1''-\frac{s_1''}{4}\right|+\left|s_1''-\frac{r_1''}{4}\right|\right\}\left|s_1''-\frac{s_1''}{4}\right|}{|r_1''-s_1''|+\left|s_1''-\frac{s_1''}{4}\right|}\right] + \frac{1}{20t}|r_1''-s_1''|$$

Then by theorem 2.1

$$d'_{pb}(r_1''{}_0,[Vr_1''{}_0]_{\alpha(r_1''{}_0)})\le v(1-tv)r$$
$$v=\frac{\left(a_1+ta_3+\frac{a_5}{2t}\right)}{\left(1-\left(a_2+ta_3+\frac{a_4}{3}\right)\right)}<\frac{1}{t}$$

then, all the conditions of theorem 2.1 is satisfied, $\exists\,0\in\overline{B_{d'_{pb}}(r_1''{}_0,r)}$. Then 0 is the fixed point of V.

Theorem 2.2 Let (Y,d'_{pb}) is complete partial b-metric space with constant $t\ge 1$. Let $V,W:Y\to F(Y)$ is fuzzy mapping and let x_0 is any arbitrary point in Y. Suppose there exists $\alpha_V(x),\alpha_W(x)\in(0,1]\ \forall\,x\in Y$ satisfying the following conditions:

$$H_{d'_{pb}}([Vx]_{\alpha_V(x)},[Wy]_{\alpha_W(y)})\le a_1 d'_{pb}(x,[Vx]_{\alpha_V(x)}) + a_2 d'_{pb}(y,[Wy]_{\alpha_W(y)}) + a_3\begin{bmatrix} d'_{pb}(x,[Wy]_{\alpha_W(y)})+ \\ d'_{pb}(y,[Vx]_{\alpha_V(x)}) \end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x,y)$$

and

$$d'_{pb}(x_0,[Vx_0]_{\alpha_V(x_0)})\le v(1-tv)r\ \forall\,x,y\in\overline{B_{d'_{pb}}(x_0,r)},\ r>0 \text{ and } tv<1,$$

where, $v=\frac{\left(a_1+a_2+2a_3t+\frac{a_4}{t}\right)}{2-(a_1+a_2+2a_3t)}$. Also, $a_i\ge 0$, *where* $i=1,2\ldots.4$ *with* $(a_1+a_2)(t+1)+t(t+1)2a_3+a_4<2$ *and* $2a_1+2a_2+4a_3+a_4\le 2$ *where* $\sum_{i=1}^4 a_i<1$. Then, there exist x^* in $\overline{B_{d'_{pb}}(x_0,r)}$ such that the common fixed point of V and W is x^*.

Proof: Let x_0 is any arbitrary point in Y s.t. $x_1\in[Vx_0]_{\alpha_V(x_0)}$. Let a sequence $\{x_w\}$ of points in X such that $x_{2l+1}\in[Vx_{2l}]_{\alpha_V(x_{2l})}, x_{2l+2}\in[Wx_{2l+1}]_{\alpha_W(x_{2l+1})}$.

First we show that $x_w\in\overline{B_{d'_{pb}}(x_0,r)}\ \ \forall n\in N$

$$d'_{pb}(x_0,x_1)=d'_{pb}(x_0,[Vx_0]_{\alpha_V(x_0)})\le v(1-tv)r<r$$
$$\Rightarrow x_1\in\overline{B_{d'_{pb}}(x_0,r)}.\ Let\ x_2,x_3,\ldots\ldots x_j\in\overline{B_{d'_{pb}}(x_0,r)}, j\in N.$$

If $j=2l+2$ *where* $l=0,1,2\ldots..\frac{l-1}{2}$.

Now, by using lemma 1.1 we get

$d'_{pb}(x_{2l+1},x_{2l+2})$

$$= H_{d'_{pb}}\left([Vx_{2l}]_{\alpha_V(x_{2l})},[Wx_{2l+1}]_{\alpha_W(x_{2l+1})}\right)$$

$$\leq a_1 d'_{pb}\left(x_{2l},[Vx_{2l}]_{\alpha_V(x_{2l})}\right) + a_2 d'_{pb}\left(x_{2l+1},[Wx_{2l+1}]_{\alpha_W(x_{2l+1})}\right) + a_3\begin{bmatrix} d'_{pb}\left(x_{2l},[Wx_{2l+1}]_{\alpha_W(x_{2l+1})}\right)+ \\ d'_{pb}\left(x_{2l+1},[Vx_{2l}]_{\alpha_V(x_{2l})}\right)\end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x_{2l},x_{2l+1})$$

$$\leq a_1 d'_{pb}(x_{2l},x_{2l+1}) + a_2 d'_{pb}(x_{2l+1},x_{2l+2}) + a_3\begin{bmatrix} d'_{pb}(x_{2l},x_{2l+2})+ \\ d'_{pb}(x_{2l+1},x_{2l+1})\end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x_{2l},x_{2l+1})$$

Then by using 4th condition of partial b-metric space, we get

$$d'_{pb}(x_{2l+1},x_{2l+2}) \leq a_1 d'_{pb}(x_{2l},x_{2l+1}) + a_2 d'_{pb}(x_{2l+1},x_{2l+2}) + a_3\begin{bmatrix} t[d'_{pb}(x_{2l},x_{2l+1})+ \\ d'_{pb}(x_{2l+1},x_{2l+2})]- \\ d'_{pb}(x_{2l+1},x_{2l+1})+ \\ d'_{pb}(x_{2l+1},x_{2l+1})\end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x_{2l},x_{2l+1})$$

$$d'_{pb}(x_{2l+1},x_{2l+2}) \leq a_1 d'_{pb}(x_{2l},x_{2l+1}) + a_2 d'_{pb}(x_{2l+1},x_{2l+2}) + a_3 t[d'_{pb}(x_{2l},x_{2l+1}) + d'_{pb}(x_{2l+1},x_{2l+2})] + \frac{a_4}{2t}d'_{pb}(x_{2l},x_{2l+1})$$

$$d'_{pb}(x_{2l+1},x_{2l+2}) \leq \left(a_1 + a_3 t + \frac{a_4}{2t}\right)d'_{pb}(x_{2l},x_{2l+1}) + (a_2 + a_3 t)d'_{pb}(x_{2l+1},x_{2l+2})$$

(4)

Also

$$d'_{pb}(x_{2l+2},x_{2l+1}) = H_{d'_{pb}}\left([Vx_{2l+1}]_{\alpha_V(x_{2l+1})},[Wx_{2l}]_{\alpha_W(x_{2l})}\right)$$

$$\leq a_1 d'_{pb}\left(x_{2l+1},[Vx_{2l+1}]_{\alpha_V(x_{2l+1})}\right) + a_2 d'_{pb}\left(x_{2l},[Wx_{2l}]_{\alpha_W(x_{2l})}\right) + a_3\begin{bmatrix} d'_{pb}\left(x_{2l+1},[Wx_{2l}]_{\alpha_W(x_{2l})}\right)+ \\ d'_{pb}\left(x_{2l},[Vx_{2l+1}]_{\alpha_W(x_{2l+1})}\right)\end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x_{2l+1},x_{2l})$$

$$d'_{pb}(x_{2l+2},x_{2l+1}) \leq a_1 d'_{pb}(x_{2l+1},x_{2l+2}) + a_2 d'_{pb}(x_{2l},x_{2l+1}) + a_3\begin{bmatrix} d'_{pb}(x_{2l+1},x_{2l+1})+ \\ d'_{pb}(x_{2l},x_{2l+2})\end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x_{2l+1},x_{2l})$$

$$d'_{pb}(x_{2l+2},x_{2l+1}) \leq a_1 d'_{pb}(x_{2l+1},x_{2l+2}) + a_2 d'_{pb}(x_{2l},x_{2l+1}) + a_3\begin{bmatrix} d'_{pb}(x_{2l+1},x_{2l+1}) \\ +t[d'_{pb}(x_{2l},x_{2l+1})+ \\ d'_{pb}(x_{2l+1},x_{2l+2})] \\ -d'_{pb}(x_{2l+1},x_{2l+1})\end{bmatrix} + \frac{a_4}{2t}d'_{pb}(x_{2l+1},x_{2l})$$

$$d'_{pb}(x_{2l+2},x_{2l+1}) \leq a_1 d'_{pb}(x_{2l+1},x_{2l+2}) + a_2 d'_{pb}(x_{2l},x_{2l+1}) + a_3 t[d'_{pb}(x_{2l},x_{2l+1}) + d'_{pb}(x_{2l+1},x_{2l+2})] + \frac{a_4}{2t}d'_{pb}(x_{2l+1},x_{2l})$$

$$d'_{pb}(x_{2l+2},x_{2l+1}) \leq (a_1 + a_3 t)d'_{pb}(x_{2l+1},x_{2l+2}) + \left(a_2 + a_3 t + \tfrac{a_4}{2t}\right)d'_{pb}(x_{2l},x_{2l+1}) \qquad (5)$$

On adding equation (4) and (5), we get

$$2d'_{pb}(x_{2l+2},x_{2l+1}) \leq \left(a_1 + a_3 t + \frac{a_4}{2t}\right)d'_{pb}(x_{2l},x_{2l+1}) + (a_1 + a_2 + 2a_3 t)d'_{pb}(x_{2l+1},x_{2l+2}) + \left(a_2 + a_3 t + \tfrac{a_4}{2t}\right)d'_{pb}(x_{2l},x_{2l+1})$$

$$2d'_{pb}(x_{2l+2},x_{2l+1}) \leq \left(a_1 + a_3 t + \frac{a_4}{2t} + a_2 + a_3 t + \frac{a_4}{2t}\right)d'_{pb}(x_{2l},x_{2l+1}) + (a_1 + a_2 + 2a_3 t)d'_{pb}(x_{2l+1},x_{2l+2})$$

$$2d'_{pb}(x_{2l+2},x_{2l+1}) \leq \left(a_1 + a_2 + 2a_3 t + \frac{a_4}{t}\right)d'_{pb}(x_{2l},x_{2l+1}) + (a_1 + a_2 + 2a_3 t)\, d'_{pb}(x_{2l+1},x_{2l+2})$$

$$2d'_{pb}(x_{2l+2},x_{2l+1}) - (a_1 + a_2 2a_3 t)\, d'_{pb}(x_{2l+1},x_{2l+2}) \leq \left(a_1 + a_2 + 2a_3 t + \frac{a_4}{t}\right)d'_{pb}(x_{2l},x_{2l+1}) + (a_1 + a_2 + 2a_3 t)\, d'_{pb}(x_{2l+1},x_{2l+2})$$

$$d'_{pb}(x_{2l+2},x_{2l+1}) \leq \frac{\left(a_1 + a_2 + 2a_3 t + \frac{a_4}{t}\right)}{2 - (a_1 + a_2 + 2a_3 t)}d'_{pb}(x_{2l},x_{2l+1}) \qquad (6)$$

As, $\nu = \frac{\left(a_1 + a_2 + 2a_3 t + \frac{a_4}{t}\right)}{2-(a_1+a_2+2a_3 t)} < \frac{1}{t}$

Then by the equation (6), we get

$$d'_{pb}(x_{2l+1},x_{2l+2}) \leq \nu d'_{pb}(x_{2l},x_{2l+1}) (7)$$

On the same way, if $j = 2l + 2$ $where$ $l = 0,1,2 \ldots\ldots \frac{l-1}{2}$.

$$d'_{pb}(x_{2l+2},x_{2l+3}) \leq \nu d'_{pb}(x_{2l+1},x_{2l+2}) \qquad (8)$$

then by eq. (7) $d'_{pb}(x_{2l+1},x_{2l+2}) \leq \nu^{2l+1} d'_{pb}(x_0,x_1)$ (9)

and by eq. (8) $d'_{pb}(x_{2l+2},x_{2l+3}) \leq \nu^{2l+2}\, d'_{pb}(x_0,x_1)$ (10)

then by adding equation (9) and (10), we have

$$d'_{pb}(x_j,x_{j+1}) \leq \nu^j d'_{pb}(x_0,x_1) \quad \forall\, j \in N \qquad (11)$$

Now,

$$d'_{pb}(x_0,x_{j+1}) \leq [t d'_{pb}(x_0,x_1) + t^2 d'_{pb}(x_1,x_2) + \cdots + t^{j+1} d'_{pb}(x_j,x_{j+1})] - d'_{pb}(x_2,x_1) - d'_{pb}(x_2,x_2) - \ldots - d'_{pb}(x_j,x_j)$$

$$d'_{pb}(x_0,x_{j+1}) \leq t d'_{pb}(x_0,x_1) + t^2 \nu d'_{pb}(x_0,x_1) + \cdots + t^{j+1}\nu^j d'_{pb}(x_0,x_1)] - \sum_{s=1}^{j} d'_{pb}(x_s,x_s)$$

Now, we show that $d'_{pb}(x_s,x_s) = 0$

Then,

$$d'_{pb}(x_s,x_s) = H_{d'_{pb}}\left([Vx_{s-1}]_{\alpha_V(x_{s-1})},[Wx_{s-1}]_{\alpha_W(x_{s-1})}\right)$$

$$\leq a_1 d'_{pb}\left(x_{s-1}, [Vx_{s-1}]_{\alpha_V(x_{s-1})}\right) + a_2 d'_{pb}\left(x_{s-1}, [Wx_{s-1}]_{\alpha_W(x_{s-1})}\right) + a_3\begin{bmatrix} d'_{pb}\left(x_{s-1}, [Wx_{s-1}]_{\alpha_W(x_{s-1})}\right) + \\ d'_{pb}\left(x_{s-1}, [Vx_{s-1}]_{\alpha_V(x_{s-1})}\right)\end{bmatrix} + \frac{a_4}{2t} d'_{pb}(x_{s-1}, x_{s-1})$$

By using 1st condition of partial b-metric

$$d'_{pb}(x_s, x_s) \leq a_1 d'_{pb}(x_{s-1}, x_s) + a_2 d'_{pb}(x_{s-1}, x_s) + a_3\begin{bmatrix} d'_{pb}(x_{s-1}, x_s) + \\ d'_{pb}(x_{s-1}, x_s)\end{bmatrix} +$$

$$d'_{pb}(x_0, x_{j+1}) \leq [t d'_{pb}(x_0, x_1) + t^2 v d'_{pb}(x_0, x_1) + \cdots \ldots\ldots + t^{j+1} v^j d'_{pb}(x_0, x_1)]$$

$$d'_{pb}(x_0, x_{j+1}) = \frac{t\left(1-(tv)^{j+1}\right)}{1-tv} d'_{pb}(x_0, x_1)$$

$$d'_{pb}(x_0, x_{j+1}) \leq \frac{t\left(1-(tv)^{j+1}\right)}{1-tv} v(1-tv) r < r$$

$\Rightarrow x_{j+1} \in \overline{B_{d'_{pb}}(x_0, r)}$. Hence by induction $x_w \in \overline{B_{d'_{pb}}(x_0, r)}\ for\ all\ w \in N$.

Now, we show that sequence $\{x_w\}$ is Cauchy sequence.

So, equation (11) can be written as

$$d'_{pb}(x_w, x_{w+1}) \leq v^w d'_{pb}(x_0, x_1)$$

Let i and w be two positive integers, $i < w$

$$d'_{pb}(x_i, x_w) \leq t\left[d'_{pb}\begin{pmatrix} x_i, x_{i+1}) + \\ d'_{pb}(x_{i+1}, x_w\end{pmatrix}\right] - d'_{pb}(x_{i+1}, x_{i+1})$$

$$d'_{pb}(x_i, x_w) \leq t d'_{pb}(x_i, x_{i+1})) + t^2 d'_{pb}(x_{i+1}, x_{i+2} + \ldots\ldots\ldots.. + t^{w-i} d'_{pb}(x_{w-1}, x_w)$$

$$d'_{pb}(x_i, x_w) \leq [t v^i d'_{pb}(x_0, x_1) + t^2 v^{i+1} d'_{pb}(x_0, x_1) + \ldots\ldots\ldots\ldots + t^{w-i} v^{w-1} d'_{pb}(x_0, x_1)]$$

$$d'_{pb}(x_i, x_w) \leq t v^i \begin{bmatrix} 1 \\ + tv + t^2 v^2 \\ + \cdots \ldots \\ + t^{w-i-1} v^{w-i-1}\end{bmatrix} d'_{pb}(x_0, x_1)$$

$$d'_{pb}(x_i, x_w) \leq \frac{t v^i}{1-vt} d'_{pb}(x_0, x_1)$$

$$\therefore\ d'_{pb}(x_i, x_w) \leq \frac{t v^i}{1-vt} d'_{pb}(x_0, x_1)$$

As $d'_{pb}(x_i, x_w) \to 0\ as\ i, w \to \infty$

Hence, the sequence $\{x_w\}$ is Cauchy sequence in $\overline{B_{d'_{pb}}(x_0, r)}$.

By completeness of $\overline{B_{d'_{pb}}(x_0, r)}$, there exist a point $x^* \in N$.

Since, $\{x_w\}$ converges to x^* as $w \to \infty$

$$\lim_{w\to\infty} d'_{pb}(x_w, x^*) = d'_{pb}(x^*, x^*) = \lim_{w\to\infty} d'_{pb}(x_w, x_w) = 0$$

$$\frac{a_4}{2t} d'_{pb}(x_{s-1}, x_s)$$

$$d'_{pb}(x_s, x_s) \leq (a_1 + a_2 + 2a_3) d'_{pb}(x_{s-1}, x_s) + \frac{a_4}{2t} d'_{pb}(x_{s-1}, x_s)$$

$$d'_{pb}(x_s, x_s) \leq \left(a_1 + a_2 + 2a_3 + \frac{a_4}{2}\right) d'_{pb}(x_{s-1}, x_s)$$

$$d'_{pb}(x_s, x_s) \leq \lambda d'_{pb}(x_{s-1}, x_s)$$

and so on

$$d'_{pb}(x_s, x_s) \leq \lambda^s d'_{pb}(x_0, x_1)$$

Where $\lambda = 2a_1 + 2a_2 + 4a_3 + a_4$

Then by taking limit $s \to \infty$, we get

$$d'_{pb}(x_s, x_s) = 0$$

Now, we need to prove x^* is the common fixed point of V and W.

$$d'_{pb}\left(x^*, [Wx^*]_{\alpha_W(x^*)}\right) \leq t\begin{bmatrix} d'_{pb}(x^*, x_{2w+1}) \\ + d'_{pb}\left(x_{2w+1}, [Wx^*]_{\alpha_W(x^*)}\right)\end{bmatrix}$$

$$\leq t[d'_{pb}(x^*, x_{2w+1}) + H_{d'_{pb}}\left([Vx_{2w}]_{\alpha_V(x_{2w})}, [Wx^*]_{\alpha_W(x^*)}\right)$$

$$\leq t\begin{bmatrix} d'_{pb}(x^*, x_{2w+1}) + \\ a_1 d'_{pb}\left(x_{2w}, [Vx_{2w}]_{\alpha_V(x_{2w})}\right) + \\ a_2 d'_{pb}\left(x_{2l+1}, [Wx^*]_{\alpha_W(x^*)}\right) + \\ a_3\begin{bmatrix} d'_{pb}\left(x_{2w}, [Wx^*]_{\alpha_W(x^*)}\right) + \\ d'_{pb}\left(x^*, [Vx_{2w}]_{\alpha_V(x_{2w})}\right)\end{bmatrix} + \\ \frac{a_4}{2t} d'_{pb}(x_{2w}, x^*)\end{bmatrix}$$

$$d'_{pb}\left(x^*, [Wx^*]_{\alpha_W(x^*)}\right) \leq t\begin{bmatrix} d'_{pb}(x^*, x_{2w+1}) + \\ a_1 d'_{pb}(x_{2w}, x_{2w+1}) + \\ a_2 d'_{pb}\left(x_{2w+1}, [Wx^*]_{\alpha_W(x^*)}\right) + \\ a_3\begin{bmatrix} d'_{pb}\left(x_{2w}, [Wx^*]_{\alpha_W(x^*)}\right) + \\ d'_{pb}(x^*, x_{2w+1})\end{bmatrix} + \\ \frac{a_4}{2t} d'_{pb}(x_{2w}, x^*)\end{bmatrix}$$

Then by taking $w \to \infty$, we get

$$[1 - t(a_2 + a_3)] d'_{pb}\left(x^*, [Wx^*]_{\alpha_W(x^*)}\right) \leq 0$$

So, $[Wx^*]_{\alpha_W(x^*)} \in x^*$

This implies that $x^* \in Y$is a fixed point of W.

On same way we can prove that x^* is the unique fixed point of V and W.

III. CONCLUSIONS

We have introduced the fuzzy mapping concept in partial b-metric space. Some fixed point theorems are proved to get the unique common fixed point. This result extends the result of Shoaib et al. [18].

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CONFLICT OF INTEREST

Author has no conflict of interest.

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